Varianta 032

Subiectul I

a) $2\sqrt{5}$. b) 1. c) a=-1, b=0. d) $\sqrt{15}$. e) a=$\frac{4\sqrt{2}}{3}$. f) $\frac{1}{6}$.

Subiectul II

1. a) 2 elemente inversabile: $\hat{1},\hat{3}$. b) soluțiile ecuației sunt $\hat{0},\hat{2}$;

c) $\hat{1}\cdot\hat{2}\cdot+\hat{2}\cdot\hat{3}+\hat{3}\cdot\hat{1}=\hat{3}$. d) 2^4=16. e) $\frac{8}{16}=\frac{1}{2}$.

2. a) x= -1 asimptotă verticală. b) $\lim_{x\to\infty}\frac{x^2+x+1}{x^2+x}$=1.

c) f'(x)=$\frac{x^2+2x}{(x+1)^2}$. d) f''(x)=$\frac{2}{(x+1)^3}\Rightarrow$f''(x)<0 pentru x$\in(-\infty,-1)\Rightarrow$ f concavă pe ($-\infty$,-1).

e) $\lim_{x\to\infty}\frac{1}{x^2}\int_0^x\left(t+\frac{1}{t+1}\right)dt=\lim_{x\to\infty}\frac{1}{x^2}\left(\frac{x^2}{2}+\ln(x+1)\right)=\frac{1}{2}+\lim_{x\to\infty}\frac{\ln(x+1)}{x^2}=\frac{1}{2}$.

Subiectul III

a) z= $w=0\Rightarrow$ f(0)=2f(0)$\Rightarrow$f(0)=0. z= w=1$\Rightarrow$f($1\cdot1$)=f(1)f(1) $\Rightarrow$f(1)^2=f(1) $\Rightarrow$f(1)=0 sau f(1)=1, dar f(1)=0 este contradicție cu f injectivă.

b) f(z_1+z_2)=f(z_1)+f(z_2) adevărat. P(n)$\to$P(n+1):
Presupunem f($z_1+z_2+\ldots+z_n$)=
=f(z_1)+f(z_2)+…+f(z_n) adevarata şi demonstrăm că
f($z_1+z_2+\ldots+z_n+z_{n+1}$)= f($z_1$)+f($z_2$)+…+f($z_n$)+f($z_{n+1}$).
Avem f($z_1+z_2+\ldots+z_n+z_{n+1}$)= f($z_1+z_2+\ldots+z_n$)+f($z_{n+1}$)= f($z_1$)+f($z_2$)+…+f($z_n$)+f($z_{n+1}$) .

c) $z_1=z_2=\ldots z_n=1\Rightarrow$f(n)=nf(1)=n$\Rightarrow$f(n)=n, oricare ar fi n$\in$ **N**.
Avem f(z+(-z))=f(z)+f(-z) $\Rightarrow$0=f(z)+f(-z) $\Rightarrow$f(-z)=-f(z) $\forall$z$\in$ **Z** $\Rightarrow$f(x)=x,$\forall$ x$\in$ **Z**.
Avem1=f(1)=f$\left(\frac{n}{n}\right)$=f$\left(\frac{1}{n}+\ldots+\frac{1}{n}\right)$=f$\left(\frac{1}{n}\right)$+…+ f$\left(\frac{1}{n}\right)$=nf$\left(\frac{1}{n}\right)$ $\Rightarrow$ f$\left(\frac{1}{n}\right)=\frac{1}{n}$,
f$\left(\frac{m}{n}\right)$= f$\left(\frac{1}{n}+\ldots+\frac{1}{n}\right)$= f$\left(\frac{1}{n}\right)$+…+ f$\left(\frac{1}{n}\right)$=mf$\left(\frac{1}{n}\right)=\frac{m}{n}$, deci f(r)= r,oricare ar fi r$\in$ **Q**.

d) Fie $x>0\Rightarrow f(x)=f(\sqrt{x}\cdot\sqrt{x})=\left(f(\sqrt{x})\right)^2>0$.

e) Fie $x_1<x_2$, $x_2-x_1>0$ $\Rightarrow$f(x_2-x_1)>f(0)$\Rightarrow$f(x_2)-f(x_1)>0$\Rightarrow$ $f(x_1)<f(x_2)$.

f) Fie $x\in\mathbf{R}\setminus\mathbf{Q}$. Pentru orice numere rationale x_1,x_2 cu $x_1<x<x_2$ avem $f(x_1)<f(x)<f(x_2)$ $\Leftrightarrow$ $x_1<f(x)<x_2$ si cum x_1,x_2 sunt arbitrare rezulta $f(x)=x$ .In concluzie, tinand seama si de c), rezulta $f(x)=x,\forall x\in\mathbf{R}$

g) $f(i\cdot i)=f(i)\cdot f(i)\Rightarrow f(-1)=(f(i))^2\Rightarrow f^2(i)=-1\Rightarrow f(i)=-i$ sau $f(i)=i$.

h) $f(z)=f(a+bi)=f(a)+f(i)\cdot f(b)\Rightarrow f(a)+if(b)=a+bi=z, f(a)-if(b)=a-bi=\bar{z}$.

Subiectul IV

a) $f_1(x)=\int_0^x f_0(t)dt=\int_0^x (t-\sin t)dt=\left.\frac{t^2}{2}\right|_0^x+\cos t\Big|_0^x=\cos x-\cos 0+\frac{x^2}{2}=\cos x+\frac{x^2}{2}-1,\ \forall x\in[0,\infty)$

b) $f_1'(x)=x-\sin x\Rightarrow f_1''(x)=1-\cos x;\ \cos x\le 1,\ \forall x\in R\Rightarrow 1-\cos x\ge 0\Rightarrow f_1''(x)\ge 0\Rightarrow f$ convexa

c) pt $n=0$ verificarea este facuta. Sa demonstram ca $P(n)\to P(n+1): f_{2n+2}(x)=\int_0^x f_{2n+1}(t)dt$

$$f_{2n+1}(x)=\int_0^x f_{2n}(t)dt=\int_0^x\left[(-1)^n\left(-\sin t+\frac{t}{1!}-\frac{t^3}{3!}+\ldots+(-1)^n\frac{t^{2n+1}}{(2n+1)!}\right)\right]dt=$$

$$=(-1)^n\left(\cos x-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\ldots+(-1)^n\frac{x^{2n+2}}{(2n+2)!}\right)\Rightarrow$$

$$f_{2n+2}(x)=\int_0^x(-1)^n\left[\cos t-1+\frac{t^2}{2!}-\frac{t^4}{4!}+\ldots+(-1)^n\frac{t^{2n+2}}{(2n+2)!}\right]dt=$$

$$=(-1)^{n+1}\left(-\sin x+\frac{x}{1!}-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+3}}{(2n+3)!}\right)$$

d) prin inductie : pentru $n=0$ este evident; $f_n(x)\ge 0\Rightarrow\int_0^x f_n(t)dt\ge 0\Rightarrow f_{n+1}(x)\ge 0$

e) $f_n(x)\ge 0\Rightarrow f_{4n}(x)\ge 0\Rightarrow(-1)^{2n}\left(-\sin x+\frac{x}{1!}-\frac{x^3}{3!}+\ldots+(-1)^{2n}\frac{x^{4n+1}}{(4n+1)!}\right)>0\Rightarrow$

$\frac{x}{1!}-\frac{x^3}{3!}+\ldots+(-1)^{2n}\frac{x^{4n+1}}{(4n+1)!}>\sin x$, $f_{2(2n-1)}(x)\ge 0\Rightarrow$ membrul stang al inegalitatii

f) din e) $\Rightarrow 0<\sin x-\frac{x}{1!}+\frac{x^3}{3!}+\ldots-\frac{x^{4n-1}}{(4n-1)!}<\frac{x^{4n+1}}{(4n+1)!}\xrightarrow{n\to\infty}0$

g) presupunem $\sin 1\in\mathbf{Q}$ si deci exista $p\in\mathbf{Z}$, $q\in\mathbf{N}^*$ asfel incat $\sin 1=\frac{p}{q}$. Din e) $\Rightarrow$

$$1-\frac{1}{3!}+\ldots-\frac{1}{(4n-1)!}<\sin 1<1-\frac{1}{3!}+\ldots-\frac{1}{(4n+1)!},\ \forall n\in\mathbf{N}$$

In particular $1-\frac{1}{3!}+\ldots-\frac{1}{(4q-1)!}<\frac{p}{q}<1-\frac{1}{3!}+\ldots-\frac{1}{(4q+1)!}\Rightarrow$

$$-\frac{1}{(4q-1)!}<\frac{p}{q}-1+\frac{1}{3!}+\ldots-\frac{1}{(4q-3)!}<0\Rightarrow-1<(4q-1)!\left(\frac{p}{q}-1+\frac{1}{3!}+\ldots-\frac{1}{(4q-3)!}\right)<0$$

Cum $(4q-1)!\left(\frac{p}{q}-1+\frac{1}{3!}+\ldots-\frac{1}{(4q-3)!}\right)\in\mathbf{Z}$ obtinem contradictie.